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Preface

> "*I dream of the day when selfishness will no longer reign in the sciences, when we will come together to study, instead of sending sealed envelopes to academics, we will hasten to publish our slightest observations as long as they are new, and we will add 'I don't know the rest'.*"
>
> - Évariste Galois

Galois Theory is considered the pinnacle of undergraduate algebra for mathematics majors. In particular, at the college level, the modern algebra syllabus is generally designed to make Galois Theory understandable in its full theoretical glory. In that context, its power is made manifest and its many applications open to further investigation.

The challenge in this monograph is to expose the inner workings of Galois Theory's theoretical foundation in a way understandable to a motivated High School student so that its stunning applications (e.g., to the factoring of n^{th} degree polynomials and ruler and compass constructions) may be accessible to one not formally trained in undergraduate level mathematics.

Let's be clear up front: this monograph is not a textbook. It is not in the traditional *definition, theorem, proof* format of a traditional undergraduate level mathematics textbook. Instead, we make assertions without formally proving them.[1] The intention is to provide a high-level conceptual overview of the relevant mathematical background and theory sufficient to appreciate and understand the main result of this particular monograph: *Why polynomials of degree* 5 *or greater are not solvable by radicals.*[2]

[1] For those interested in formal mathematical rigor the proofs of all assertions can be found in any decent undergraduate level Abstract Algebra textbook. See [Pin82] for a thorough, well written reference.

[2] Not even by Galois himself. ☹

The primary focus is to investigate the solvability of the general polynomial equation of degree n

$$x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0 = 0$$

where the coefficients $\{a_i\}$ are rational numbers.

In this context, *solvability* means finding the roots of the above equation as functions of the coefficients a_i. That is, if $x = r_i$ is a solution to the above polynomial, can it be expressed as a function of the coefficients, that is $r_i = f(a_0, a_1, ..., a_{n-1})$? Furthermore, can it be so expressed using only basic arithmetic operations (addition, subtraction, multiplication, and division) and the extraction of radicals (e.g., n^{th} roots). This was the goal of classical algebra since antiquity. Galois Theory explains why that goal is unattainable for polynomials of degree 5 and greater.

It is well know that one can indeed solve linear ($n = 1$), quadratic ($n = 2$), cubic ($n = 3$), and quartic ($n = 4$) polynomials in full generality, expressing the roots as functions of the coefficients. (This is illustrated in the Appendices for cubic and quartic polynomials.)

But not for the quintic ($n = 5$)! That's where *Radicals Go To Die.*

Chapter 1 recapitulates the important historical context that led to Galois' tremendous insight.

Chapter 2 presents the mathematical foundations that support the subsequent analysis. This presentation is expository but hopefully sufficient for an understanding of said analysis.

Chapter 3 is the meat of this monograph. It explains why radicals die at the quintic and beyond.

Chapter 4 presents unexpected applications of Galois Theory. In particular, a number of unsolved *ruler and compass* constructions from antiquity are shown to be impossible using the algebraic machinery presented herein. Leave it to Galois to help put that conundrum to rest. If only the ancients knew! So many frustrating and wasted century-hours would have been saved.

An Epilogue concludes the discussion with some closing remarks.

The Appendices contain supplemental material for the interested reader. Appendix A[Bri19a] details the derivation of the roots for the general cubic (degree 3) polynomial. Appendix B[Bri19b] does the same for the quartic (degree 4) polynomial.

These two appendices were taken from a most wonderful website belonging to Professor Curtis Bright, who has the distinction of having an Erdös Number of 2. Let's hope his web site remains active so that the reader may enjoy it. At that site you will also find the full expansion of the quartic formula that yields the roots of the general quartic equation. Before you pass on, you must experience this extraordinary piece of mathematical kitsch.