Preface

It strikes this author that there is an inherent danger in condensing complex subject matter, that has a deep and complicated history, into a sharply focused piece of expository writing. The danger is that the intellectual evolution that preceded the subject matter is lost to the reader. This loss of historical context may make the subject matter appear as if by magic, without a satisfying sense of historical context: how and why such inquiries originally evolved and what motivated the mind that gave them birth.

The author does not want budding young Mathematicians to conclude erroneously that the subject matter covered herein is a trivial side-show, consisting of the (mathematical) historical equivalent of sleight of hand or smoke and mirrors. This is very much not the case. Perhaps a bit of historical background will help here.

Georg Cantor [1845-1918], a famous 19th century German mathematician, is the creator of transfinite set theory and the theory of transfinite numbers. In a metaphorical sense, Georg Cantor is 'Mr. Infinity' with respect to mathematics. Everything covered in this monograph traces back to his ground breaking work. Cantor culminated his life's work in a two volume set [1895 and 1897] referred to as his *Beiträge*.

Early in his career, Cantor made historically important contributions in Numerical Analysis and Functional Analysis; the latter with respect to the representation of continuous functions by trigonometric series. But it was his related investigations into the *continuum* - the real number line - that spun him off in directions that forever changed the course of mathematical analysis.

Cantor remains one of the most imaginative and controversial figures in the long history of mathematics. He was controversial mostly because his work created a major upheaval in mathematics during its day. In fact, some of his peers[1] attempted to prohibit him from publishing his work on Transfinite Set Theory! Such was their fear of its disruption to Mathematics as a discipline.[2]

Luckily, Cantor's detractors lost their philosophical assault on what they viewed as heretical mathematics. No less a giant than David Hilbert endorsed infinite mathematical objects as legitimate objects of study.[3] Cantor's influence survives to this day as his research not only established an entirely new field of mathematics, but it also has deep importance in the fields of topology, number theory, analysis, the theory of functions, as well as the entire field of modern logic.

[1] Including his former professor in Berlin - Leopold Kronecker - who praised Cantor's early work in Number Theory.

[2] Kronecker, for example, believed the only legitimate mathematical objects are *finite* in nature. Quantification over infinite domains was not considered meaningful by him. This mathematical philosophy was referred to as *finitism*.

[3] Hilbert referred to infinite collections as *ideal* objects, to distinguish them from (finite) *concrete* objects.

A good starting point for readers new to transfinite numbers is *The Philosophy of Set Theory - An Historical Introduction to Cantor's Paradise* by Mary Tiles. This work targets mathematics and philosophy majors, but is accessible to others.

For a fascinating and comprehensive discussion of Cantor's work - his mathematics and philosophy of the infinite, including the mathematical antecedents that lead him to it - see the definitive volume by Joseph Warren Dauben.[Dau79] This work is polymorphic, all at once: biographical, historical, mathematical, and even psychological in its breadth. It contains a wonderful epilogue on the significance of Cantor's personality. The psychological profile is quite informative and enlightening. Warning: much of this content is advanced.

For an English translation of Cantor's two famous Beiträge, see Philip E. B. Jourdain's excellent volume.[Jou17] This work includes a remarkably comprehensive Introduction that traces the lineage from Fourier, Cauchy, Weierstrass, Dedekind, Dirichlet, Riemann and Hankel to Cantor. Warning: this content is advanced.

Of course, while you are waiting for the delivery of the above two text books, be sure to consult the Wikipedia page for Cantor.[Wikc]

Prologue

What does *Infinity* mean to you?

Ask most people and they will respond with a vague answer, such as "*something without end*" or "*something unbounded*"; and that will be the extent of their understanding.[4]

Now, as far it goes, those statements are not wrong. They possess a kernel of truth. But they are incomplete and inadequate explanations of what turns out to be a quite deep and, at times, a maddeningly paradoxical concept in Mathematics.

As *homo sapiens*, we evolved over millions of years in *finite* surroundings. Our entire experience with the external world involved physical boundaries and limits. Our senses and intellect were likewise constrained, since there were no selective pressures to conceive of or care about "The Infinite". It simply had no bearing on the selective pressures that governed life - and its evolution - on Earth.

Because of this evolutionary fact, we do not possess a natural intuition for the infinite. Without intellectual effort, certain logical conclusions regarding constructs of infinite extent appear paradoxical, if not nonsensical. One's first instinct may be to dismiss paradoxical conclusions as "foolish reasoning" or "not worthy of pursuit". However, careful thought reveals subtly beautiful explanations and helps expand our consciousness in surprising ways.

In this monograph, several fascinating properties of Infinity are investigated in the context of Mathematics. No formal mathematical training beyond Algebra and Analytic Geometry is assumed. That said, the reader will be exposed to topics and concepts not covered in the typical high school curriculum. Chapter 1 has been written to bridge any gaps the reader may have in approaching this paper. Any motivated Senior High School student should be comfortable and intellectually prepared to wrestle with this exploration of the infinite. There is nothing discussed herein above the grasp of an exceptionally motivated and gifted grade-school student who possesses strong critical thinking skills.

The topics covered here are meant as supplementary self-study for motivated grade-school students who welcome an introduction to mathematical topics and concepts usually not encountered until college and usually only within a Math Major curriculum.

Mathematics is a broad field of study. For the intellectually curious, its depth and breadth are, for all practical purposes, infinite! So much so, that there is almost certainly no one alive today who can claim to be a master of it all. As with all of modern science, deep technical dives into niche areas of inquiry can take a lifetime to explore. Advances come slowly and are often the result of team efforts and much collaboration. Expert fluency in Mathematics is not a trivial pursuit.

[4] Well, that is fair enough. Infinity is certainly not your typical day to day concern.

All that said, it remains true that even at relatively superficial levels, mathematics can reveal insights about the world around us and even philosophical musings that continue to astound. Exposure to such knowledge is never a waste of one's time. Stay curious and grow!

It is an unfortunate truth that the beauty and insight revealed in Mathematics does not truly reveal itself until one has had exposure to it beyond the standard grade school curriculum, which all to often can be stultifying. It is hoped that exposure to fascinating topics outside of the typical grade school curriculum will stimulate further interest in mathematics and reveal the unimaginably wide world of mathematics that exists beyond grade-school arithmetic, algebra, and plane geometry.

This monograph, the first of an intended set of many such, is an attempt to kindle such interest by revealing some surprising consequences that result from even a superficial investigation of Infinity.

Here is a summary of the contents of this monograph:

Chapter 1 covers the background needed to fully comprehend the main body of the monograph. This includes important analytic and geometric concepts referenced throughout the text. Read this chapter carefully and make sure you understand it before continuing further.

Chapter 2 discusses many properties of Sets: determining their cardinality (size), how to combine them, and the use of bijective maps to compare cardinalities across sets. The importance of the set of natural numbers - in the quest to help categorize the infinite cardinalities - a set is also discussed.

Chapter 3 introduces *countably infinite* collections and their isomorphic relationships with the Natural Numbers ($\mathbb{N}$). We show that the Integers ($\mathbb{Z}$) and Rationals ($\mathbb{Q}$) have the same cardinality. We introduce the befuddling Ramanujan Sum to show how operations involving infinity can lead to very confusing results if not handled with care.

Chapter 4 introduces *uncountably infinite* sets; one of them being the familiar set of Real Numbers ($\mathbb{R}$). We demonstrate that the cardinality of this set is far larger than that of the previous countably-infinite collections. That is, there exists a higher-order of infinity than "countably infinite". In fact, there are infinitely many infinite cardinalities! We also introduce some astonishing transformations across uncountably-infinite sets and marvel at the "elasticity' of the notion of *length* under these transformations.

Chapter 5 covers the crowning-jewel of Infinity paradoxes: *The Banach-Tarski Paradox.* Discover the mathematical basis for the story of the *Fishes and Loaves*! Discover how a 3-D Ball can be decomposed into parts and reassembled as two balls with *each of the two balls having the same volume as the original*! [What!?] You don't want to miss it!

Chapter 6 concludes with some final thoughts about our introductory exploration of The Infinite.

Contents