Preface

This monograph is the second in a series of such works addressed to the advanced high school student who has an interest in being exposed to mathematical ideas and concepts that are not a part of a standard high school mathematics curriculum. The intent is to expose the student to the enormous breadth and beauty of mathematics, much of which is lost to a curriculum narrowly focused on preparing students for standardized tests. The hope is that this exposure will excite and encourage the student to view the discipline with an enhanced appreciation.

The first monograph, *A Finite Introduction To Infinity*, introduced the reader to transfinite numbers: an infinity of infinities. Using simple mathematical concepts, the monograph guided the user through distinctions such as *countable* versus *uncountable* infinities and related such magnitudes to various sets, such as the sets of Integers, Rationals, and Real numbers and an extraordinary construct referred to as Cantor Dust. It then launched into its culmination: The Banach-Tarksi Paradox. It was a wild, and hopefully, enjoyable exposition into an infinitely fascinating topic [pun intended].

In this volume we investigate various way to count. Yes...*count*.

Please resist the urge to stop reading right here and now! Be open to the possibility that, as with Infinity, your preconceived ideas about what *counting* means may be incomplete. For example, in the first monograph, *A Finite Introduction To Infinity* we discovered that adding new elements to an infinite set does not necessarily increase its size! This notion of counting required the explicit establishment of a *bijection* in order to quantify the 'count' (e.g., cardinality) of a set and to compare cardinalities across sets.

For some applications, counting requires one to distinguishing between differing characteristics among the elements under consideration. For example, consider a drawer full of socks of different colors where the socks are randomly distributed. To ask questions such as '*What is the probability of randomly picking 2 matching socks with just two attempts*' or '*How many different ways are there to randomly select 3 red socks in 5 attempts?*' involves 'counting', but it must be carried out with special care. As another example, consider the card game *Poker*, in particular 5-card poker. To determine the odds of being dealt a particular poker-hand, say 3 of a kind, one must *count* the number of ways to construct such a hand and divide that number by the number of all possible hands (which requires its own count).

There are special techniques for 'counting' in these and other like situations. In this monograph, we discuss and investigate a number of techniques that facilitate the ability to *count* in the context of complex scenarios, perhaps constrained by certain assumptions.

Through the selection of interesting problem domains, the reader will be introduced to advanced mathematical topics generally not encountered outside of a college level mathematics program; such as Stochastic Processes and Queueing Theory. Although the conceptual ideas will be new and challenging, the mathematical machinery supporting these concepts should be familiar and accessible to the interested high school student.

Prologue

Most thoughful people very much *wish* to count. And, by and large, they *should* count! But the truly important question is: are they *able* to count, followed closely by *how* do they count?

What does *Counting* mean to you?

Let's begin by restating, just for the record, something to which you may have been exposed back in grammar school; namely so-called *The Five Principles of Counting*:

1. `Stable Order` - Being able to verbalize the act of counting in proper order.
2. `One-to-One Correspondence` - Understanding each counted item is associated with one, and only one, number.
3. `Cardinality` - Understanding that a a counting sequence reveals the quantity for the set of things being counted.
4. `Abstraction` - Understanding that it does not matter what is being counted. Their characteristics do not matter. How one counts does not change.
5. `Order Irrelevance` - The order in which items are counted is irrelevant.

Now some clarification of the above principles.

First of all, these five principles are used by teachers as part of a developmental progression - i.e., roadmap - to derive a teaching plan.[1]

The first three principles relate to the 'HOW' of counting. There is nothing really controversial about them.

However, Principles 4 and 5 refer to the 'WHAT' of counting. Depending on the actual application, in more advanced mathematics (e.g., not grade/pre-school introductory counting) these two principles are definitely not valid universally. In this monograph we will see examples where characteristics (e.g., color, rank) and order (of choice) can and do matter with respect to counting outcomes.

In this monograph, *counting* is used as a convenient short-hand label to encapsulate the many different ways in which one may carry out enumerative processes on diverse collections of elements in an attempt to quantify some desired characteristic.

[1]Based on the work of Gelman & Galistel (1978). Their seminal work overturned the Piagetian orthodoxy concerning the ontogenesis of the capacity (in children) to represent natural numbers. See https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3856691/.

Yes, the author is aware of the danger that the above characterization may be obscure to the point of meaninglessness. But hopefully, dear reader, you can appreciate its well-crafted generality!

The point is that techniques for 'counting' can differ depending on the problem domain under consideration. Hopefully, the reader will find the selected problem domains interesting and that they convey that serious counting involves more than fingers and toes.

Chapter 1 presents some preliminary mathematical machinery that comes up in later chapters. Much of this material will likely be new to the intended audience but its basis is high school level algebra, kicked up a level with respect to conceptualization.

Chapter 2 introduces out first problem domain: *Poker*. Using the combinatoric counting methods reviewed in Chapter 1, we compute the probabilities of all of the possible 5-card poker hands that may be dealt from a well-shuffled standard deck of 52 playing cards.

Chapter 3 introduces counting in the context of uncertainty. In this chapter a Tennis game is modeled as a special kind of Stochastic process, called a Markov Process. We then derive, as a nifty closed-form polynomial equation, the probability of a player winning a tennis game. The techniques used here involve the aggregation of combinatoric calculations and the application (properly interpreted) of some matrix algebra introduced in Chapter 1. Using the results of this analysis, we extend it to compute the probability of winning a Set and a Match. The results are tabulated, parameterized by various initial point-probability assumptions.

Chapter 4 introduces more advanced stochastic processes - Pure-Birth and Birth-Death processes - which are the basis of Queuing Theory and its many applications. We investigate the 'flows' represented by the associated mathematical models and derive some important equations: the Poisson Distribution equation and a steady-state equilibrium probability distribution. These equations allow us to associate probabilities with assumed population counts in the system. These are examples of 'probabilitic counting'.

Chapter 5 introduces in detail another counting technique involving the translation of the original problem space from its *space/time* domain to it *frequency* domain equivalent. The mathematical model is solved in the *frequency* domain and the answer re-translated back into the original *space/time* domain. Partial-fraction decomposition is demonstrated here as well. This entire bit of *mathemagical* sleight-of-hand is truly a black-art when it works!

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